KÄHLER POLARISATION OF HAMILTONIAN SYSTEMS SUBJECT TO SECOND CLASS CONSTRAINTS

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Abstract. The necessary and sufficient conditions are established for the second-class constraint surface to be (an almost) Kähler manifold. The deformation quantisation for such systems is scetched resulting in the Wick-type symbols for the respective Dirac brackets.

1. Introduction

The progress of the quantum field theory was always related, directly or indirectly, with in-depth study of quantisation methods. Recent years have brought the explosive developments in the Deformation Quantization theory (see [1] for review and further references). By the optimistic expectations an appropriate adaptation of this method to the field theory would help to resolve some old problems concerning the non-perturbative quantisation of essentially non-linear models (like the non-linear sigma-models on a homogeneous background and, in particular, strings on AdS space), when any linear approximation can violate fundamental symmetries of the theory. To provide a natural frame for a particle interpretation of the fields quantised, the Wick-type deformation quantisation seems to be most appropriate (see [2], [3], [4] an references therein).

Any actual application of these methods to the field theory should also be supplemented by a proper account of the Hamiltonian constraints, which are characteristic for all practically interesting field theory models. The first step in this direction was done in [5], where the BRST version [6] of the Fedosov deformation quantization [7] was extended to the second-class constraint systems. The next step would consist in generalising the construction of the paper [5] (giving the Weyl type symbols for the second class systems) to the Wick symbol calculus on a non-linear phase-space subject to the Hamiltonian constraints. As the Wick structure is usually inherited from the Kähler structure, one naturally comes to consideration of the Hamiltonian reduction on the (almost-)Kähler manifolds. In this note we study, at first, the question of the classical Hamiltonian reduction by second class-constraints and examine the compatibility of this reduction with the Kähler structure of the enveloping manifold. The extension of these results to the quantum case is briefly discussed in the concluding section, more details will be given elsewhere.

2. Almost-Kähler manifolds

In this section we briefly recall some basic definitions and facts concerning the geometry of almost-Kähler manifolds. For more details see, for example, [8].

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The almost-Kähler manifold (M, J, ω), is a real $2n$-dimensional manifold M together with an almost-complex structure J and a symplectic form ω which are compatible in the following sense:

$$\omega(JX, JY) = \omega(X, Y) \tag{2.1}$$

for any vector fields X, Y. In other words, the smooth field of authomorphisms

$$J : TM \to TM, \qquad J^2 = -1 \tag{2.2}$$

is a canonical transformation of the tangent bundle w.r.t. symplectic structure ω. Then $g(X, Y) = \omega(JX, Y)$ is J-invariant (pseudo-)Riemannian metric on M,

$$g(JX, JY) = g(X, Y). \tag{2.3}$$

The almost-complex structure J splits the complexified tangent bundle $T^{\mathbb{C}}M$ onto two transverse mutually conjugated sub-bundles: $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$, such that

$$J_p X = iX, \qquad \forall X \in T_p^{(1,0)}M, \tag{2.4}$$

$$J_p Y = -iY, \qquad \forall Y \in T_p^{(0,1)}M, \tag{2.5}$$

for every $p \in M$. In the natural frames $\partial_i = \partial/\partial x^i$, dx^j associated to local coordinates (x^i) on M we have $J^i_j = J(dx^i, \partial_j)$, $\omega_{ij} = \omega(\partial_i, \partial_j)$, $g_{ij} = g(\partial_i, \partial_j)$, and

$$J^i_j = g^{ik}\omega_{kj} = g_{jk}\omega^{ki}, \tag{2.6}$$

where (g^{ik}) and (ω^{ik}) are the inverse matrices to (g_{ij}) and (ω_{ij}), respectively. It is well known that any symplectic manifold (M, ω) admits a compatible almost-complex structure J, which turns it to an almost-Kähler manifold.

Alternatively, the almost-Kähler manifold can be defined as a pair (M, Λ) in which M is $2n$-dimensional real manifold equipped with a degenerate Hermitian form Λ, such that

$$\mathrm{rank}\Lambda = \frac{1}{2}\dim M = n, \tag{2.7}$$

$$\det(\mathrm{Im}\Lambda) \neq 0. \tag{2.8}$$

The equivalence of both the definitions is set by the formula

$$\Lambda_{jk} = g_{jk} + i\omega_{jk}. \tag{2.9}$$

The vector fields of type $(1,0)$ or $(0,1)$ w.r.t. the almost-complex structure J, being defined by relations (2.4), (2.5), are nothing but the right/left null-vectors for the form Λ.

Now let ∇ be the unique torsion-free connection respecting metric g. This connection is known to respect the symplectic structure ω whenever (M, J, ω) is the Kähler manifold (equivalently, iff J is integrable), so that tensor

$$T^i_{jk} = \omega^{in}\nabla_n\omega_{jk}, \tag{2.10}$$

vanishes. In general case one can always define a new affine connection $\overline{\nabla}$, which already respects both the metric and symplectic structures, by adding an appropriate torsion. The canonical possibility is to put $\overline{\nabla} = \nabla + T$. Indeed,

$$\overline{\nabla}_i\omega_{jk} = \nabla_i\omega_{jk} - T^n_{ij}\omega_{nk} - T^n_{ik}\omega_{nj} = \tag{2.11}$$

$$= \nabla_i \omega_{jk} + \nabla_j \omega_{ki} + \nabla_k \omega_{ij} = (d\omega)_{ijk} = 0.$$

This choice for $\overline{\nabla}$ is known as the Lichnerowicz connection [9] of an almost-Kähler manifold. Thus, the existence of a torsion-free connection respecting Hermitian form Λ is the necessary and sufficient condition for the almost-Kähler manifold M to be a Kähler one. Introduce the Hermitian form $\Lambda^{ik} = g^{ik} + i\omega^{ik}$ on the complexified cotangent bundle of M. In view of the identity $\Lambda^{ij}\Lambda_{jk} = 0$ and the rank condition (2.7) any vector field X of type $(0,1)$ can be (non-uniquely) represented as

$$X = Z_j \Lambda^{ji} \partial_i \tag{2.12}$$

for some one-form $Z = Z_i(x)dx^i$. Commuting two vector fields of the form (2.12) and requiring the result to annihilate the form Λ (2.9) on the left we get the integrability condition for the distribution $T^{(0,1)}M$:

$$(\Lambda^{in}\partial_n \Lambda^{jk} - \Lambda^{in}\partial_n \Lambda^{jk})\Lambda_{km} = 0 \tag{2.13}$$

Similarly, the subbundle $T^{(1,0)}M$ is integrable iff

$$(\Lambda^{ni}\partial_n \Lambda^{jk} - \Lambda^{ni}\partial_n \Lambda^{jk})\Lambda_{mk} = 0. \tag{2.14}$$

(Here we use of the identity $\Lambda_{ki}\Lambda^{ij} = 0$.)

Surprisingly, the Lichnerowicz torsion T being associated to the metric g and the symplectic form ω can be written in terms of the almost-complex structure J along. After some algebra one may find that

$$T^i_{jk} = -\frac{1}{4}N^i_{jk}, \quad N^i_{jk} = J^n_k \partial_n J^i_j - J^n_j \partial_n J^i_k + J^i_n \partial_j J^n_k - J^i_n \partial_k J^n_j \tag{2.15}$$

The tensor N is known as Nijenhuis tensor of an almost-complex structure J. It is $N \neq 0$ which is the only obstruction for the structure J to be integrable.

¿From the view point of symplectic geometry, the integrable holomorphic/anti-holomorphic distributions $T^{(1,0)}M$ and $T^{(0,1)}M$ define a pair of transverse Lagrangian polarizations of M, i.e. $\omega|_{T^{(1,0)}M} = \omega|_{T^{(0,1)}M} = 0$. The existence of such polarizations is of primary importance for the physical applications as it makes possible to define the notion of *physical state* for a quantum-mechanical system, at least in the framework of geometric quantization [10].

3. Hamiltonian reduction by second-class constraints.

Consider (M, J, ω) as the phase-space of a mechanical system with the Poisson bracket

$$\{f, g\} = \omega^{ij}\partial_i f \partial_j g, \qquad f, g \in C^\infty(M),$$

and let θ_a be a set of $2m$, $(m < n)$ smooth functions on M such that the matrix of Poisson brackets

$$\omega_{\alpha\beta} = \{\theta_\alpha, \theta_\beta\} \tag{3.1}$$

is non-degenerate on the whole phase space, $\det(\omega_{\alpha\beta}) \neq 0$. Then equations

$$\theta_\alpha = 0, \qquad \alpha = 1, ..., 2m, \tag{3.2}$$

called second-class constraints, extract $2(n-m)$-dimensional smooth surface $\Sigma \subset M$. It is the surface where the dynamics of the mechanical system is assumed to evolve[1]. As the matrix (3.1) is nowhere degenerate, the pair $(\Sigma, \omega|_\Sigma)$ is a symplectic manifold again, $\omega|_\Sigma$ is a restriction of two-form ω onto surface Σ. Denote by $\{\cdot,\cdot\}_\Sigma$ the corresponding Poisson bracket. According to the general theory of second-class constraint systems the Poisson bracket $\{\cdot,\cdot\}_\Sigma$ can be extended to a bracket on the whole manifold M. This extension, known as Dirac bracket, is given by

$$\begin{aligned}\{f,g\}_D &= \{f,g\} - \{f,\theta_\alpha\}\omega^{\alpha\beta}\{\theta_\beta, g\} = \widetilde{\omega}^{ij}\partial_i f \partial_j g, \\ \widetilde{\omega}^{ij} &= \omega^{ij} - \omega^{in}\partial_n\theta_\alpha\omega^{\alpha\beta}\partial_m\theta_\beta\omega^{mj}\end{aligned} \tag{3.3}$$

where $(\omega^{\alpha\beta})$ is the inverse matrix for $(\omega_{\alpha\beta})$. Since the rank of Poisson bi-vector $\widetilde{\omega}$ is constant, rel. (3.3) defines a regular Poisson bracket on M, having θ's as the Casimir functions, i.e.

$$\{f,\theta_\alpha\}_D = \{\theta_\alpha, f\}_D = 0, \qquad \forall f \in C^\infty(M). \tag{3.4}$$

As any regular Poisson manifold, $(M, \{\cdot,\cdot\}_D)$ has a symplectic foliation: the surfaces of constant values of the Casimir functions are symplectic submanifolds with symplectic structures induced by ω. In particular, the constraint surface Σ is a symplectic leaf corresponding to zero locus of the Casimir functions.

We would like to emphasize that the Dirac bracket construction strongly depends on the particular choice of the constraint functions, not just on the embedding $\Sigma \subset M$ itself. Taking another constraint basis

$$\theta_\alpha \to \theta'_\alpha = A_\alpha^\beta(x)\theta_\beta, \qquad \det(A_\alpha^\beta) \neq 0, \tag{3.5}$$

which defines the same submanifold Σ by imposing the equations $\theta'_\alpha = 0$, one gets a different extension for the Poisson bracket $\{\cdot,\cdot\}_\Sigma$, which, however, coincides with (3.3) on the constraint surface Σ.

An important advantage of the Dirac bracket as compared to the bracket $\{\cdot,\cdot\}_\Sigma$ on the reduced phase space is that it allows to describe the constrained dynamics without explicit solving constraints (the latter may be rather problematic, especially in the field-theoretical models). Another way to deal with the second-class constraints (which probably is more suitable for the subsequent quantisation) is to convert them into the first class ones by extending the original phase space. For this end consider the direct product $\mathcal{M} = M \times \mathbb{R}^{2m}$ equipped with the symplectic form [5]

$$\Omega = \omega_{ij}dx^i \wedge dx^j + d\eta^\alpha \wedge d\theta_\alpha, \tag{3.6}$$

where η^α are linear coordinates on $\mathbb{R}^{2m}$. Two-form Ω is obviously closed and nondegenerate. The corresponding Poisson bracket $\{\cdot,\cdot\}_\mathcal{M}$, being determined by the inverse matrix to Ω in the local coordinates $(\xi^A) = (x^i, \eta^\alpha)$, reads

$$\Omega^{-1} = (\Omega^{AB}) = \begin{pmatrix} \widetilde{\omega}^{ij} & -\omega^{il}\partial_l\theta_\gamma\omega^{\gamma\beta} \\ \omega^{jl}\partial_l\theta_\gamma\omega^{\gamma\alpha} & \omega^{\alpha\beta}\end{pmatrix} \tag{3.7}$$

[1]Note, that the usual definition of the second-class constraints requires the invertibility of the matrix (3.1) only on the constraint surface (3.2) and hence, in some it's tubular neighborhood. However, it is not clear at the moment whether it is possible to perform the consistent (algebraic, geometric, deformation, ...) quantisation under such a weakened condition without explicit solving the constraints.

As is seen the upper left block of Ω^{-1} is nothing but the Dirac bi-vector (3.3). So, the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ coincides with the Dirac one when is evaluated on η-independent functions [2]. In particular, the constraints θ_α, considered as the functions on the extended phase space $\mathcal{M}$, turn out to be in involution,

$$\{\theta_\alpha,\theta_\beta\}_{\mathcal{M}}=0 \tag{3.8}$$

The Hamiltonian reduction of $(\mathcal{M},\{\cdot,\cdot\}_{\mathcal{M}})$ w.r.t. the first-class constraints θ_α immediately leads to the dynamics on the constrained phase space $(\Sigma,\{\cdot,\cdot\}_\Sigma)$. In so doing, the equations

$$\eta^\alpha=0 \tag{3.9}$$

may be thought of as admissible gauge fixing conditions for the first-class constraints θ_α,

$$\{\theta_\alpha,\eta^\beta\}=\delta^\beta_\alpha \tag{3.10}$$

To put it differently, eqs. (3.2), (3.9), taken together, define the second-class theory on the extended phase space $\mathcal{M}$. This extended constrained dynamics is equivalent to the original second class constraint system on M.

Denote by $\Lambda|_\Sigma$ the restriction of the Hermitian form Λ onto the constraint surface Σ. The main question we address in this section is in the following: Given a second-class constraint system (M,Λ,θ_a), under which conditions the (almost)-Kähler structure Λ on M induces a Kähler structure $\Lambda|_\Sigma$ on the constraint surface $\Sigma:\theta=0$?

To avoid having to deal with the degenerate structure of the Dirac bracket we will perform all the calculations on the extended phase space $\mathcal{M}=M\times\mathbb{R}^{2m}$ and then reinterpret the results in terms of inner geometry of M. To begin with, we endow $\mathcal{M}$ with the (pseudo-)Riemannian metric $G=G_{AB}d\xi^Ad\xi^B$ of the form

$$(G_{AB})=\begin{pmatrix} g_{ij} & \partial_i\theta_\alpha \\ \partial_j\theta & 0\end{pmatrix} \tag{3.11}$$

The inverse metric reads

$$G^{-1}=(G^{AB})=\begin{pmatrix} \widetilde{g}^{ij} & g^{il}\partial_l\theta_\gamma g^{\gamma\beta} \\ g^{jl}\partial_l\theta_\gamma g^{\gamma\alpha} & -g^{\alpha\beta}\end{pmatrix} \tag{3.12}$$

$$\widetilde{g}^{ij}=g^{ij}-g^{in}\partial_n\theta_\alpha g^{\alpha\beta}\partial_m\theta_\beta g^{mj}$$

As will be seen bellow, the direct product structure of the extended phase-space, being considered in combination to the invariance of Ω and G under η-translations, allows to identify various geometric structures on M with certain blocks of geometric structures on $\mathcal{M}$.

Introduce the pair of Hermitian matrices

$$W=G+i\Omega \tag{3.13}$$

$$\lambda=(\lambda_{\alpha\beta}),\qquad \lambda_{\alpha\beta}=\partial_i\theta_\alpha\Lambda^{ij}\partial_j\theta_\beta \tag{3.14}$$

The next proposition provides robust criteria for the constraint surface to be a Kähler manifold.

[2]This observation offers the most simple proof of the Jacobi identity for the Dirac bracket: it takes place because Ω. is closed.

Proposition. *With above notations and definitions the following statements are equivalent:*

i) $(\Sigma, \Lambda|_\Sigma)$ *is a Kähler manifold;*

ii) $(\mathcal{M}, W)$ *is a Kähler manifold;*

iii) there exist a (complex) basis of constraints θ_α *and the holomorphic coordinates on* M *in which the matrix* $\Pi = (\partial_i\theta_\alpha)$ *takes the block-diagonal form*

$$\Pi = \begin{pmatrix} \pi^+ & 0 \\ 0 & \pi^- \end{pmatrix}, \tag{3.15}$$

where $\pi^\pm$ *are* $n \times m$ *matrices;*

iv) $\mathrm{rank}(\lambda) = m$ *and*

$$T_{ijk} = \nabla_i\omega_{jk} - g^{\alpha\beta}\nabla_i\partial_k\theta_\beta P_j^m\partial_m\theta_\alpha - (j \leftrightarrow k) = 0, \tag{3.16}$$

where ∇ *is the torsion-free connection respecting metric* g *and* $P_j^i = \delta_j^i + J_j^i$.

Sketch of the proof. Since the imaginary parts of both the Hermitian matrices $\Lambda|_\Sigma$ and W are non-degenerate the first two statements hold iff

$$\mathrm{rank}\Lambda|_\Sigma = \frac{1}{2}\dim\Sigma = n - m, \qquad \mathrm{rank}W = \frac{1}{2}\dim\mathcal{M} = n + m, \tag{3.17}$$

provided the right and the left kernel distributions of the forms are integrable. The proof of equivalence of these conditions to each other and to the algebraic parts of iii) and iv) is a simple exercise in linear algebra.

Now assuming the rank conditions (3.17) to be satisfied let us examine the integrability of the almost-Kähler structures W and $\Lambda|_\Sigma$. As was mentioned in the previous section this is equivalent to vanishing the respective Lichnerowicz torsion (2.10). Consider first the case of the extended phase-space $(\mathcal{M}, W)$. Let D be the torsion-free connection respecting metric $G = G_{AB}d\xi^A d\xi^b$ and let $\widetilde{\Gamma}^A_{BC}$ be the corresponding Christoffel symbols. Splitting the coordinates ξ^A as (x^i, η^α) we find only two nonvanishing components of the Christoffel symbols,

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{in}\partial_n\theta_\alpha\widetilde{\Gamma}^\alpha_{jk}, \qquad \widetilde{\Gamma}^\alpha_{ij} = -g^{\alpha\beta}\nabla_i\partial_j\theta_\beta, \tag{3.18}$$

where $\nabla = \partial + \Gamma$ is the torsion-free connection respecting g. Then the only nonzero component of the Lichnerowicz torsion $T_{ABC} = \Omega_{AN}T^N_{BC} = D_A\Omega_{BC}$ is given by

$$\begin{aligned} T_{ijk} &= D_i\Omega_{jk} = \nabla_i\omega_{jk} + \Gamma^\alpha_{ij}\Omega_{\alpha k} + \Gamma^\alpha_{ik}\Omega_{j\alpha} = \\ &= \nabla_i\omega_{jk} + \Gamma^\alpha_{ik}P_j^m\partial_m\theta_\alpha - (j \leftrightarrow k) \end{aligned} \tag{3.19}$$

Note that for the Kähler manifold (M, Λ) the first term in rel. (3.19) vanishes. Due to the rank condition (3.17) the form W has exactly $m + n$ left null-vectors

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + v^\alpha(x)\frac{\partial}{\partial\eta^\alpha}, \tag{3.20}$$

and $2n$ of them are obviously of the form $\partial/\partial\eta^\alpha$. So, there are $n - m$ additional null-vectors of the form $Y = \xi^i\partial_i$ with ξ's satisfying

$$\xi^i\Lambda_{ij} = 0, \qquad \xi^i\partial_i\theta_\alpha = 0 \tag{3.21}$$

The last equation implies that the vector Y (i) is tangent to the constraint surface Σ and (ii) it is annihilated by Λ and therefore by $\Lambda|_\Sigma$. These $n-m$ left null-vectors span a half of complexified tangent bundle $T^{\mathbb{C}}\Sigma$. The complimentary distribution of the right null-vectors is obtained by the complex conjugation. The integrability of both the distributions immediately follows from equation (3.21) by the Frobenius theorem.□

The two nonzero components (3.18) of the Christoffel symbols $\widetilde{\Gamma}^A_{BC}$ admit a straightforward interpretation in terms of inner geometry of M. Namely, $\widetilde{\Gamma}^i_{jk}$ are the Christoffels of the unique connection $\widetilde{\nabla}$ which preserves both the degenerate symmetric tensor $\widetilde{g}^{ij}$ (the upper left block of the inverse metric G^{-1} (3.12)) and its null-covectors $d\theta_\alpha$. Adding to $\widetilde{\Gamma}$ the torsion $T^i_{jk} = \widetilde{\omega}^{in}T_{njk}$ (3.19) we get the covariant derivative $\widetilde{\nabla}' = \widetilde{\nabla} + T$, which is continuing to respect $\widetilde{g}^{ij}$ and $d\theta_\alpha$ and it also preserves the Dirac bi-vector $\widetilde{\omega}^{ij}$. As to the M-tensors $\widetilde{\Gamma}^\alpha_{ij}$, $\alpha = 1, ..., 2m$, they can be identified with the exterior curvatures of the surface Σ as they characterize the embedding of Σ into the Riemann manifold (M, g). Indeed, when $M = \mathbb{R}^{2n}$ and g is flat, the equations $\widetilde{\Gamma}^\alpha_{ij} = 0$ describe $2(n-m)$-dimensional planes in $\mathbb{R}^{2n}$.

4. Concluding remarks

In this note we study the Hamiltonian reduction of an (almost-)Kähler manifold by second-class constraints. The criteria are found providing the reduced dynamics to be still of a Kähler structure.

The main technique point of our approach is the conversion of the second-class constraint system to a first-class one in the extended phase space $\mathcal{M} = M \times \mathbb{R}^{2m}$, being a direct product of the original phase manifold to the linear space which dimension equals to the number of the second-class constraints. This space is naturally endowed with the special symplectic and metric structures invariant under translations along $\mathbb{R}^{2m}$. It turns out that the Kähler geometry of the reduced phase space (being given implicitly until one keeps the constraints unresolved) is completely encoded, and may be studied, by means of the explicit Kähler structure on the extended phase space $\mathcal{M}$.

Although $\mathcal{M}$ has appeared as a suitable auxiliary construction to study the classical Hamiltonian reduction its role becomes even more important upon the quantization. Let us take a quick look at this point. When $\mathcal{M}$ is the Kähler manifold it can be easily quantised (applying the construction of the paper [3]) to produce the algebra of Wick symbols being an associative deformation in $\hbar$ of the ordinary commutative algebra of functions on $\mathcal{M}$. In so doing, the invariance of the Kähler structure under the $\mathbb{R}^{2m}$-shifts is transformed into the respective symmetry of the Wick star-product. As the result, the functions, being constant on $\mathbb{R}^{2m}$, form a closed subalgebra, and thus the Wick star-product on $\mathcal{M}$ induces that on the original phase-space M. The latter star product is characterized by the nontrivial central subalgebra generated by the second-class constraints (which are Casimir functions of the respective Dirac bracket) and it is compatible with the Wick polarization. It is almost obvious that the classical limit of the corresponding star-commutator on M would be nothing but the classical Dirac bracket. In such a manner, the deformation quantization on $\mathcal{M}$ gives rise to what one may naturally call as *quantum Dirac brackets of Wick type.*

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